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ON THE GREEN'S FUNCTIONS FOR TWO-PHASE TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIA

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Abstract—This paper is concerned with the problem of point force and point charge applied in the interior of an infinite two-phase transversely isotropic piezoelectric solid. Based on the general solutions, by using the method of the image source, a series of displacement functions are constructed. The Green's functions are obtained when arbitrary constants are determined by the boundary conditions on the interface. Furthermore, we reduce the present solutions to the extension of Mindlin results and of Lorentz results for semi-infinite transversely isotropic piezoelectric materials by suitable substitutions of boundary conditions on the interface. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

In recent years, intelligent or smart structures and systems have become an emerging new research area. Piezoelectric material, due to its characteristic direct-converse piezoelectric effect, has naturally received considerable attention. The fundamental solutions or the Green's functions can be used to construct many analytical solutions of practical importance. They are also very important in the areas of study such as point defects, inclusion particles in materials and, especially, the boundary element method.

For isotropic infinite media, the point force solutions are just the well known Kelvin results. With regards to the transversely isotropic solids, Pan and Chou (1976) have made key contributions to the point force solutions for an infinite body. They obtained the solutions for both the two different cases of the eigenvalues of materials by using a complicated general solution. Much earlier than this, Hu (1956) had also obtained the point force solutions for an infinite transversely isotropic solids. Regarding the study of Green's functions for piezoelectric media, Chen (1993) and Chen and Lin (1993) expressed the infinite body Green's functions and their derivatives of first and second degree as the contour integrals over the unit circle by using three-dimensional Fourier transforms. Dunn (1994) gave an explicit solution for the Green's functions for an infinite transversely isotropic piezoelectric solid by taking Radon transforms, coordinator transformation and evaluation of residues in sequence. Regarding the plane problem, Lee and Jiang (1994) have obtained a fundamental solution for an infinite plane by using double Fourier transforms.

The point force solutions for an isotropic half-space had been obtained by Mindlin (1936) and have been widely referenced. The formula and developing processes of the Mindlin solutions and their applications can be found in the publications of Lure (1964) and Brebbia (1984). Phan-Thien (1983) showed that Lorentz's result could be further extended to the problem of a point force applied in the interior of a half-space involving a fixed plane boundary. Pan and Chou (1979a) had obtained the Green's functions for a transversely isotropic half-space for both the two cases of eigenvalues. With regard to piezoelectric media, Wang and Chen (1994) had studied the problem of concentrated forces normal to the plane of isotropy applied at the boundary of a transversely isotropic piezoelectric half-space. Wang and Zheng (1995) had obtained the solutions to the problem of concentrated forces parallel to the plane of isotropy applied at the boundary of the piezoelectric half-space. Sosa and Castro (1994) had obtained the solutions to the problem of concentrated loads at the boundary of a piezoelectric half-plane. Rongved (1955) and Huang and Wang (1991) studied the problem of a point force applied in one of two bonded semi-infinite isotropic solids. Dundurs and Hetenyi (1963) extended the study to two semi-infinite isotropic solids in smooth contact. With regard to the transversely isotropic materials, Pan and Chou (1979b) studied the Green's functions for two-phase transversely isotropic materials and obtained closed-form solutions for all the four cases the eigenvalues of the materials might happen to satisfy. Other works concerning this problem can be found in the publications referenced in those papers. However, for transversely isotropic piezoelectric media, either for a two-phase media or for a half-space media, the study of the problem to point force and point charge applied in the interior of the medium has not been published. This paper, based on the general solution, by using the method of the image source, systematically studies the problem of point force and point charge applied in the interior of an infinite two-phase transversely isotropic piezoelectric solid, and obtained overall solutions. Furthermore, we obtained the extension results of Mindlin and of Lorentz for a piezoelectric half-space.

For future reference, we quoted the linear constitutive relations for transversely isotropic piezoelectric media as follows:

$$\begin{aligned}
\sigma_x &= c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial \phi}{\partial z}\\
\sigma_y &= c_{12}\frac{\partial u}{\partial x}+c_{11}\frac{\partial v}{\partial y}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial \phi}{\partial z}\\
\sigma_z &= c_{13}\frac{\partial u}{\partial x}+c_{13}\frac{\partial v}{\partial y}+c_{33}\frac{\partial w}{\partial z}+e_{33}\frac{\partial \phi}{\partial z}\\
\tau_{yz} &= c_{44}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+e_{15}\frac{\partial \phi}{\partial y}\\
\tau_{xz} &= c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+e_{15}\frac{\partial \phi}{\partial x}\\
\tau_{xy} &= c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\
D_x &= e_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\varepsilon_{11}\frac{\partial \phi}{\partial x}\\
D_y &= e_{15}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)-\varepsilon_{11}\frac{\partial \phi}{\partial x}\\
D_z &= e_{31}\frac{\partial u}{\partial x}+e_{31}\frac{\partial v}{\partial y}+e_{33}\frac{\partial w}{\partial z}-\varepsilon_{33}\frac{\partial \phi}{\partial z}
\end{aligned} \tag{1}$$

where $\sigma_i(\tau_{ij})$, D_i, $u(v, w)$ and ϕ are the components of stress, electric displacement, mechanical displacement and electric potential, respectively; c_{ij}, e_{ij} and ε_{ij} are the elastic stiffness, piezoelectric and dielectric constants, respectively.

According to Wang and Zheng (1995) and Ding *et al.* (1996), there are general solutions of the coupled equations for the transversely isotropic piezoelectric media as follows:

$$u = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial x} - \frac{\partial \psi_0}{\partial y}$$

$$v = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial y} + \frac{\partial \psi_0}{\partial x} \quad (s_1 \neq s_2 \neq s_3 \neq s_1)$$

$$w_m = \sum_{i=1}^{3} \alpha_{im} \frac{\partial \psi_i}{\partial z_i}, \quad (m = 1, 2), \tag{2}$$

where s_1, s_2, s_3 are the three roots of the characteristic equation defined in Ding *et al.* (1966), w_1 is the displacement w and w_2 is the electric potential ϕ; furthermore, functions ψ_i satisfy

$$\left(\Lambda + \frac{\partial^2}{\partial z_i^2}\right)\psi_i = 0, \quad (i = 0, 1, 2, 3) \tag{3}$$

in which $\Lambda = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, $z_i = s_i z$ and $s_0^2 = c_{66}/c_{44}$.

The coefficients α_{im} are given by

$$\alpha_{i1} = \frac{c_{11}\varepsilon_{11} - m_3 s_i^2 + c_{44}\varepsilon_{33}s_i^4}{(m_1 - m_2 s_i^2)s_i}$$

$$\alpha_{i2} = \frac{c_{11}\varepsilon_{15} - m_4 s_i^2 + c_{44}\varepsilon_{33}s_i^4}{(m_1 - m_2 s_i^2)s_i} \tag{4}$$

where

$$m_1 = \varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31})$$

$$m_2 = \varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31})$$

$$m_3 = c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2$$

$$m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31}). \tag{5}$$

Substituting eqn (2) into eqn (1), the expressions of stress and electric displacement are given in terms of functions ψ_i:

$$\sigma_x = -(c_{11} - c_{12})\frac{\partial^2 \psi_0}{\partial x \partial y} + \sum_{i=1}^{3}\left[\xi_i \frac{\partial^2}{\partial z_i^2} + (c_{11} - c_{12})\frac{\partial^2}{\partial x^2}\right]\psi_i$$

$$\sigma_y = (c_{11} - c_{12})\frac{\partial^2 \psi_0}{\partial x \partial y} + \sum_{i=1}^{3}\left[\xi_i \frac{\partial^2}{\partial z_i^2} + (c_{11} - c_{12})\frac{\partial^2}{\partial y^2}\right]\psi_i$$

$$\tau_{xy} = c_{66}\left(\frac{\partial^2 \psi_0}{\partial z_0^2} + 2\frac{\partial^2 \psi_0}{\partial x^2}\right) + 2c_{66}\sum_{i=1}^{3}\frac{\partial^2 \psi_i}{\partial x \partial y}$$

$$\tau_{xm} = -\omega_{0m}\frac{\partial^2 \psi_0}{\partial y \partial z_i} + \sum_{i=1}^{3}\omega_{im}\frac{\partial^2 \psi_i}{\partial x \partial z_i}$$

$$\tau_{ym} = \omega_{0m}\frac{\partial^2 \psi_0}{\partial x \partial z_i} + \sum_{i=1}^{3}\omega_{im}\frac{\partial^2 \psi_i}{\partial y \partial z_i}$$

$$\sigma_m = \sum_{i=1}^{3}\vartheta_{im}\frac{\partial^2 \psi_i}{\partial z_i^2} \tag{6}$$

where $m = 1, 2$; σ_1, σ_2, τ_{x1}, τ_{x2}, τ_{y1} and τ_{y2} represent σ_z, D_z, τ_{xz}, D_x, τ_{yz} and D_y, and the coefficients ξ_i, ω_{im}, ϑ_{im} are given by

$$\begin{aligned}
\xi_i &= (c_{13}\alpha_{i1}+e_{31}\alpha_{i2})s_i-c_{12}\\
\omega_{01} &= c_{44}s_0, \quad \omega_{02} = e_{15}s_0\\
\omega_{i1} &= c_{44}(s_i+\alpha_{i1})+e_{15}\alpha_{i2}\\
\omega_{i2} &= e_{15}(s_i+\alpha_{i1})-\varepsilon_{11}\alpha_{i2}\\
\vartheta_{i1} &= (c_{33}\alpha_{i1}+e_{33}\alpha_{i2})s_i-c_{13}\\
\vartheta_{i2} &= (e_{33}\alpha_{i1}-\varepsilon_{33}\alpha_{i2})s_i-e_{31}.
\end{aligned} \tag{7}$$

With regard to the problem of a point force or point charge applied in the interior of a two-phase transversely isotropic piezoelectric solid with the interface parallel to the plane of isotropy, a Cartesian coordinate system is chosen such that the xy-plane lies in the interface and the point force or point charge is acting at the point $(0, 0, h)$. Assume that the two half-spaces are perfectly bonded, thus we have the boundary conditions on the interface $(z = 0)$:

$$u = u', \quad v = v', \quad w_m = w'_m \tag{8a}$$

$$\sigma_z = \sigma'_z, \quad \tau_{xz} = \tau'_{xz}, \quad \tau_{yz} = \tau'_{yz}, \quad D_z = D'_z \tag{8b}$$

where $()'$ refers to the variables in the half-space $z \leqslant 0$ and the other ones refer to those in the other half-space $z \geqslant 0$.

In the following discussion, the displacements and electric potential are expressed as the sum of two terms, giving

$$\begin{aligned}
u &= u_1+u_2\\
v &= v_1+v_2 \quad (z \geqslant 0)\\
w_m &= w_{m1}+w_{m2}, \quad (m = 1,2)
\end{aligned} \tag{9}$$

where u_1, v_1 and w_{m1} correspond to the point force or point charge solutions for an infinite piezoelectric solids; u_2, v_2 and w_{m2} are the terms of superposition in order to satisfy the boundary conditions.

For future reference, we introduce a series of denotations:

$$\begin{aligned}
&z_i = s_i z, \quad h_i = s_i h, \quad z'_i = s'_i z,\\
&z_{ij} = z_i+h_j, \quad R_{ij} = \sqrt{x^2+y^2+z_{ij}^2},\\
&\bar{z}_{ij} = z_i-h_j, \quad \bar{R}_{ij} = \sqrt{x^2+y^2+\bar{z}_{ij}^2}, \quad (i,j = 0,1,2,3)\\
&z'_{ij} = z'_i-h_j, \quad R'_{ij} = \sqrt{x^2+y^2+z'^2_{ij}}.
\end{aligned} \tag{10}$$

2. SOLUTIONS TO THE PROBLEM OF COMBINATION OF POINT FORCE P IN z DIRECTION AND POINT CHARGE Q

This is an axisymmetric problem. Assume

$$\psi_0 = 0, \quad \psi_i = A_i\,\mathrm{sign}(z-h)\ln(\bar{R}_{ii}+s_i|z-h|), \quad (i = 1,2,3) \tag{11}$$

where $\bar{R}_{ii} = \sqrt{r^2+\bar{z}_{ii}^2}$, $r^2 = x^2+y^2$, and A_i $(i = 1,2,3)$ are arbitrary constants subject to determination.

Substituting eqn (11) into (2) and (6) yields

$$u_1 = \operatorname{sign}(z-h) \sum_{i=1}^{3} \frac{A_i x}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)} \tag{12}$$

$$v_1 = \operatorname{sign}(z-h) \sum_{i=1}^{3} \frac{A_i y}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)} \tag{13}$$

$$w_{m1} = \sum_{i=1}^{3} \frac{\alpha_{im} A_i}{\bar{R}_{ii}} \tag{14}$$

$$\sigma_{x1} = (c_{11}-c_{12}) \operatorname{sign}(z-h) \sum_{i=1}^{3} A_i \left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)} - \frac{x^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)} - \frac{x^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2} \right] - \sum_{i=1}^{3} \xi_i A_i \frac{\bar{z}_{ii}}{\bar{R}_{ii}^3} \tag{15}$$

$$\sigma_{y1} = (c_{11}-c_{12}) \operatorname{sign}(z-h) \sum_{i=1}^{3} A_i \left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)} - \frac{y^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)} - \frac{y^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2} \right] - \sum_{i=1}^{3} \xi_i A_i \frac{\bar{z}_{ii}}{\bar{R}_{ii}^3} \tag{16}$$

$$\tau_{xy1} = -2c_{66} xy \operatorname{sign}(z-h) \sum_{i=1}^{3} A_i \left[\frac{1}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)} + \frac{1}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2} \right] \tag{17}$$

$$\tau_{xm1} = -\sum_{i=1}^{3} \omega_{im} \frac{A_i x}{\bar{R}_{ii}^3} \tag{18}$$

$$\tau_{ym1} = -\sum_{i=1}^{3} \omega_{im} \frac{A_i y}{\bar{R}_{ii}^3} \tag{19}$$

$$\sigma_{m1} = -\sum_{i=1}^{3} \frac{\vartheta_{im} A_i \bar{z}_{ii}}{\bar{R}_{ii}^3}. \tag{20}$$

Consideration of the continuity of displacements u_1, v_1 and stress σ_{x1}, σ_{y1}, τ_{xy1} on $z = h$ yields

$$\sum_{i=1}^{3} A_i = 0. \tag{21}$$

Then, on the neighborhood of the plane of $z = h$, we cut an elastic layer by using two planes of $z = h \pm \varepsilon$. Consideration of the equilibrium of the elastic layer yields two additional equations:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\sigma_{z1}(x, y, h+\varepsilon) - \sigma_{z1}(x, y, h-\varepsilon)] \, dx \, dy + P = 0 \tag{22}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [D_{z1}(x, y, h+\varepsilon) - D_{z1}(x, y, h-\varepsilon)] \, dx \, dy - Q = 0. \tag{23}$$

Substituting eqn (20) into eqns (22) and (23), respectively, yields

$$4\pi \sum_{i=1}^{3} \vartheta_{i1} A_i - P = 0 \tag{24}$$

$$4\pi \sum_{i=1}^{3} \vartheta_{i2} A_i + Q = 0. \tag{25}$$

Solving the algebra eqns (21), (24) and (25), the constants A_i $(i = 1, 2, 3)$ can be determined:

$$\begin{aligned}
A_1 &= [P(\vartheta_{22}-\vartheta_{32})+Q(\vartheta_{21}-\vartheta_{31})]/b_1 \\
A_2 &= [P(\vartheta_{32}-\vartheta_{12})+Q(\vartheta_{31}-\vartheta_{11})]/b_1 \\
b_1 &= 4\pi[(\vartheta_{11}-\vartheta_{31})(\vartheta_{22}-\vartheta_{32})-(\vartheta_{21}-\vartheta_{31})(\vartheta_{12}-\vartheta_{32})] \\
A_3 &= -A_1-A_2 = [P(\vartheta_{12}-\vartheta_{22})+Q(\vartheta_{11}-\vartheta_{21})]/b_1.
\end{aligned} \tag{26}$$

Assume

$$\psi_0 = 0, \quad \psi_i = \sum_{j=1}^{3} A_{ij} \ln(R_{ij}+z_{ij}), \quad (i = 1, 2, 3). \tag{27}$$

Substituting eqn (27) into (1) and (6) yields u_2, v_2, w_{m2}, stresses and electric displacements. By adding these to the corresponding ones of eqns (12)–(20), according to eqn (9), we have

$$\begin{aligned}
u &= \sum_{i=1}^{3}\left[\operatorname{sign}(z-h)\frac{A_i x}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)}+\sum_{j=1}^{3}\frac{A_{ij}x}{R_{ij}(R_{ij}+z_{ij})}\right] \\
v &= \sum_{i=1}^{3}\left[\operatorname{sign}(z-h)\frac{A_i y}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)}+\sum_{j=1}^{3}\frac{A_{ij}y}{R_{ij}(R_{ij}+z_{ij})}\right] \\
w_m &= \sum_{i=1}^{3}\alpha_{im}\left(\frac{A_i}{\bar{R}_{ii}}+\sum_{j=1}^{3}\frac{A_{ij}}{R_{ij}}\right) \\
\tau_{xm} &= -x\sum_{i=1}^{3}\omega_{im}\left(\frac{A_i}{\bar{R}_{ii}^3}+\sum_{j=1}^{3}\frac{A_{ij}}{R_{ij}^3}\right) \\
\tau_{ym} &= -y\sum_{i=1}^{3}\omega_{im}\left(\frac{A_i}{\bar{R}_{ii}^3}+\sum_{j=1}^{3}\frac{A_{ij}}{R_{ij}^3}\right) \\
\sigma_m &= -\sum_{i=1}^{3}\vartheta_{im}\left(\frac{A_i\bar{z}_{ii}}{\bar{R}_{ii}^3}+\sum_{j=1}^{3}\frac{A_{ij}z_{ij}}{R_{ij}^3}\right)
\end{aligned} \tag{28}$$

where $m = 1, 2$, A_i are known and A_{ij} are nine arbitrary constants subject to determination by boundary conditions.

In the half-space of $z \leqslant 0$, assume

$$\psi'_0 = 0, \quad \psi'_i = \sum_{j=1}^{3} A'_{ij} \ln(R'_{ij}-z'_{ij}), \quad (i = 1, 2, 3). \tag{29}$$

Substitution of eqn (29) into (2) and (6), respectively, yields the displacements, electric potential, stresses and electric displacements as follows:

$$u' = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{A'_{ij}x}{R'_{ij}(R'_{ij}-z'_{ij})}, \quad \tau'_{xm} = x\sum_{i=1}^{3}\omega'_{im}\sum_{j=1}^{3}\frac{A'_{ij}}{R'^{3}_{ij}}$$

$$v' = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{A'_{ij}y}{R'_{ij}(R'_{ij}-z'_{ij})}, \quad \tau'_{ym} = y\sum_{i=1}^{3}\omega'_{im}\sum_{j=1}^{3}\frac{A'_{ij}}{R'^{3}_{ij}}$$

$$w'_m = -\sum_{i=1}^{3}\alpha'_{im}\sum_{j=1}^{3}\frac{A'_{ij}}{R'_{ij}}, \quad \sigma'_m = \sum_{i=1}^{3}\vartheta'_{im}\sum_{j=1}^{3}\frac{A'_{ij}z'_{ij}}{R'^{3}_{ij}}. \tag{30}$$

It follows from the boundary conditions on the interface stated as eqn (8) that

$$-A_i + \sum_{j=1}^{3} A_{ji} = \sum_{j=1}^{3} A'_{ji} \tag{31}$$

$$\alpha_{im}A_i + \sum_{j=1}^{3}\alpha_{jm}A_{ji} = -\sum_{j=1}^{3}\alpha'_{jm}A'_{ji} \tag{32}$$

$$-\omega_{i1}A_i - \sum_{j=1}^{3}\omega_{j1}A_{ji} = \sum_{j=1}^{3}\omega'_{j1}A'_{ji} \tag{33}$$

$$\vartheta_{im}A_i - \sum_{j=1}^{3}\vartheta_{jm}A_{ji} = -\sum_{j=1}^{3}\vartheta'_{jm}A'_{ji} \tag{34}$$

where $i = 1, 2, 3$ and $m = 1, 2$ and α'_{im}, ω'_{im}, ϑ'_{im}, $(i = 1,2,3,4,5, m = 1,2)$ are defined the same as eqns (4) and (7) except that the material constants are replaced by the corresponding ones in the half-space of $z \leqslant 0$, and A'_{ij}, A_{ij} $(i,j = 1,2,3)$ are arbitrary constants that can be readily determined by the above algebra equations with eqns (31)–(34).

3. SOLUTIONS TO THE PROBLEM OF POINT FORCE T IN x DIRECTION

Assume

$$\psi_0 = \frac{D_0 y}{\bar{R}_{00}+s_0|z-h|}$$

$$\psi_i = \frac{D_i x}{\bar{R}_{ii}+s_i|z-h|}, \quad (i = 1,2,3) \tag{35}$$

where D_i $(i = 0,1,2,3)$ are arbitrary constants subject to determination. Substitution of eqn (35) into (2) and (6) yields

$$u_1 = -D_0\left[\frac{1}{\bar{R}_{00}+s_0|z-h|} - \frac{y^2}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)^2}\right] + \sum_{i=1}^{3} D_i\left[\frac{1}{\bar{R}_{ii}+s_i|z-h|} - \frac{x^2}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2}\right] \tag{36}$$

$$v_1 = -\frac{D_0 xy}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)^2} - xy\sum_{i=1}^{3}\frac{D_i}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2} \tag{37}$$

$$w_{m1} = -\operatorname{sign}(z-h)x\sum_{i=1}^{3}\alpha_{im}\frac{D_i}{\bar{R}_i(\bar{R}_i+s_i|z-h|)} \tag{38}$$

$$\sigma_{x1} = (c_{11}-c_{12})D_0 x\left[\frac{1}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)^2}-\frac{2y^2}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^3}\right.$$
$$\left.-\frac{y^2}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)^2}\right]+x\sum_{i=1}^{3}D_i\left\{\frac{\xi_i}{\bar{R}_{ii}^3}-(c_{11}-c_{12})\left[\frac{3}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2}\right.\right.$$
$$\left.\left.-\frac{2x^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^3}-\frac{x^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)^2}\right]\right\} \tag{39}$$

$$\sigma_{y1} = -(c_{11}-c_{12})D_0 x\left[\frac{1}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)^2}-\frac{2y^2}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^3}\right.$$
$$\left.-\frac{y^2}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)^2}\right]+x\sum_{i=1}^{3}D_i\left\{\frac{\xi_i}{\bar{R}_{ii}^3}-(c_{11}-c_{12})\left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2}\right.\right.$$
$$\left.\left.-\frac{2y^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^3}-\frac{y^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)^2}\right]\right\} \tag{40}$$

$$\tau_{xy1} = c_{66}D_0 y\left[\frac{1}{\bar{R}_{00}^3}-\frac{2}{\bar{R}_{00}(\bar{R}_{00}+s_o|z-h|)^2}+\frac{4x^2}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^3}\right.$$
$$\left.+\frac{2x^2}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)^2}\right]-2c_{66}y\sum_{i=1}^{3}D_i\left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2}-\frac{2x^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^3}\right.$$
$$\left.-\frac{x^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)^2}\right] \tag{41}$$

$$\tau_{xm1} = \omega_{0m}\,\mathrm{sign}(z-h)D_0\left[\frac{1}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)}-\frac{y^2}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)}\right.$$
$$\left.-\frac{y^2}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^2}\right]-\mathrm{sign}(z-h)\sum_{i=1}^{3}\omega_{im}D_i\left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)}\right.$$
$$\left.-\frac{x^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)}-\frac{x^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2}\right] \tag{42}$$

$$\tau_{ym1} = \omega_{0m}\,\mathrm{sign}(z-h)D_0 xy\left[\frac{1}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)}+\frac{1}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^2}\right]$$
$$+\mathrm{sign}(z-h)xy\sum_{i=1}^{3}\omega_{im}D_i\left[\frac{1}{\bar{R}_{ii}^3(R_{ii}+s_i|z-h|)}+\frac{1}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2}\right] \tag{43}$$

$$\sigma_{m1} = \sum_{i=1}^{3}\vartheta_{im}\frac{D_i x}{\bar{R}_{ii}^3}. \tag{44}$$

The consideration of the continuity of the components w_m, τ_{xm} and τ_{ym} on $z = h$ yields

$$\sum_{i=1}^{3} \alpha_{im} D_i = 0, \quad (m = 1,2) \tag{45}$$

$$\omega_{0m} D_0 + \sum_{i=1}^{3} \omega_{im} D_i = 0, \quad (m = 1,2). \tag{46}$$

Substituting the expression of ω_{im} as shown in eqn (7) into eqn (46) and further using eqn (45), the two equations of (46) all become

$$\sum_{i=1}^{3} s_i D_i = 0. \tag{47}$$

Still, considering the equilibrium of an elastic layer cut out by two planes of $z = h \pm \varepsilon$ on the neighborhood of the plane of $z = h$ yields an additional equation, giving

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\tau_{xz1}(x,y,h+\varepsilon) - \tau_{xz1}(x,y,h-\varepsilon)]\, dx\, dy + T = 0. \tag{48}$$

Substituting eqn (42) into (48) yields

$$2\pi c_{44} s_0 D_0 - 2\pi \sum_{i=1}^{3} \omega_{i1} D_i + T = 0. \tag{49}$$

By solving eqns (45), (47) and (49) the constants D_i ($i = 0, 1, 2, 3$) can be determined, giving

$$\begin{aligned} D_0 &= -T/(4\pi c_{44} s_0) \\ D_1 &= (\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22})T/b_2 \\ D_2 &= (\alpha_{31}\alpha_{12} - \alpha_{11}\alpha_{32})T/b_2 \\ D_3 &= (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})T/b_2 \\ b_2 &= 4\pi c_{44}[s_1(\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22}) + s_2(\alpha_{31}\alpha_{12} - \alpha_{11}\alpha_{32}) + s_3(\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})]. \end{aligned} \tag{50}$$

Assume

$$\begin{aligned} \psi_0 &= \frac{D_{00} y}{R_{00} + z_{00}} \\ \psi_i &= \sum_{j=1}^{3} \frac{D_{ij} x}{R_{ii} + z_{ij}}, \quad (i = 1,2,3). \end{aligned} \tag{51}$$

Substitution of eqns (51) into (2) and (6) yields u_2, v_2, w_{m2}, stresses and electric displacements. By adding these to the corresponding ones of eqns (36)–(44) according to eqn (9), we have

$$u = -D_0 \left[\frac{1}{\bar{R}_{00} + s_0|z-h|} - \frac{y^2}{\bar{R}_{00}(\bar{R}_{00} + s_0|z-h|)^2} \right]$$

$$-D_{00} \left[\frac{1}{R_{00} + z_{00}} - \frac{y^2}{R_{00}(R_{00} + z_{00})^2} \right] + \sum_{i=1}^{3} \left\{ D_i \left[\frac{1}{\bar{R}_{ii} + s_i|z-h|} - \frac{x^2}{\bar{R}_{ii}(\bar{R}_{ii} + s_i|z-h|)^2} \right] \right.$$

$$+\sum_{j=1}^{3} D_{ij}\left[\frac{1}{R_{ij}+z_{ij}}-\frac{x^2}{R_{ij}(R_{ij}+z_{ij})^2}\right]\Bigg\}$$

$$v = -\frac{D_0 xy}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)^2}-\frac{D_{00}xy}{R_{00}(R_{00}+z_{00})^2}-xy\sum_{i=1}^{3}\left[\frac{D_i}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)^2}\right.$$

$$\left.+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}(R_{ij}+z_{ij})^2}\right]$$

$$w_m = -x\sum_{i=1}^{3}\alpha_{im}\left[\operatorname{sign}(z-h)\frac{D_i}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)}+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}(R_{ij}+z_{ij})}\right]$$

$$\tau_{xm} = \omega_{0m}\Bigg\{\operatorname{sign}(z-h)D_0\left[\frac{1}{\bar{R}_{00}(\bar{R}_{00}+s_0|z-h|)}-\frac{y^2}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)}\right.$$

$$\left.-\frac{y^2}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^2}\right]+D_{00}\left[\frac{1}{R_{00}(R_{00}+z_{00})}-\frac{y^2}{R_{00}^3(R_{00}+z_{00})}\right.$$

$$\left.-\frac{y^2}{R_{00}^2(R_{00}+z_{00})^2}\right]\Bigg\}-\sum_{i=1}^{3}\omega_{im}\Bigg\{\operatorname{sign}(z-h)D_i\left[\frac{1}{\bar{R}_{ii}(\bar{R}_{ii}+s_i|z-h|)}\right.$$

$$\left.-\frac{x^2}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)}-\frac{x^2}{\bar{R}_{ii}^2(\bar{R}_{ii}+s_i|z-h|)^2}\right]$$

$$+\sum_{j=1}^{3}D_{ij}\left[\frac{1}{R_{ij}(R_{ij}+z_{ij})}-\frac{x^2}{R_{ij}^3(R_{ij}+z_{ij})}-\frac{x^2}{R_{ij}^2(R_{ij}+z_{ij})^2}\right]\Bigg\}$$

$$\tau_{ym} = \omega_{0m}xy\Bigg\{\operatorname{sign}(z-h)D_0\left[\frac{1}{\bar{R}_{00}^3(\bar{R}_{00}+s_0|z-h|)}+\frac{1}{\bar{R}_{00}^2(\bar{R}_{00}+s_0|z-h|)^2}\right]$$

$$+D_{00}\left[\frac{1}{R_{00}^3(R_{00}+z_{00})}+\frac{1}{R_{00}^2(R_{00}+z_{00})^2}\right]\Bigg\}+xy\sum_{i=1}^{3}\omega_{im}\Bigg\{\operatorname{sign}(z-h)D_i$$

$$\times\left[\frac{1}{\bar{R}_{ii}^3(\bar{R}_{ii}+s_i|z-h|)}+\frac{1}{\bar{R}_{ii}^2(R_{ii}+s_i|z-h|)^2}\right]+\sum_{j=1}^{3}D_{ij}\left[\frac{1}{R_{ij}^3(R_{ij}+z_{ij})}+\frac{1}{R_{ij}^2(R_{ij}+z_{ij})^2}\right]\Bigg\}$$

$$\sigma_m = x\sum_{i=1}^{3}\vartheta_{im}\left(\frac{D_i}{R_{ii}^3}+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}^3}\right) \tag{52}$$

where $m = 1, 2$; D_0, D_i $(i = 1,2,3)$ are known and D_{00}, D_{ij} $(i,j = 1,2,3)$ are 10 arbitrary constants subject to determination by boundary conditions.

In the half-space of $z \leqslant 0$, assume

$$\psi_0' = \frac{L_{00}'y}{R_{00}'-z_{00}'};\quad \psi_i' = \sum_{j=1}^{3}\frac{L_{ij}'x}{R_{ij}'+z_{ij}'},\quad (i = 1,2,3) \tag{53}$$

Substitution of eqn (53) into (2) and (6), respectively, yields the displacements, electric potential, stresses and electric displacements as follows:

$$u' = -L_{00}'\left[\frac{1}{R_{00}'-z_{00}'}-\frac{y^2}{R_{00}'(R_{00}'-z_{00}')^2}\right]+\sum_{i=1}^{3}\sum_{j=1}^{3}L_{ij}'\left[\frac{1}{R_{ij}'-z_{ij}'}-\frac{x^2}{R_{ij}'(R_{ij}'-z_{ij}')^2}\right]$$

$$v' = -\frac{L_{00}'xy}{R_{00}'(R_{00}'-z_{00}')^2}-xy\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{L_{ij}'}{R_{ij}'(R_{ij}'-z_{ij}')^2}$$

$$w'_m = x\sum_{i=1}^{3}\alpha'_{im}\sum_{j=1}^{3}\frac{L'_{ij}}{R'_{ij}(R'_{ij}-z'_{ij})}$$

$$\tau'_{xm} = -\omega'_{0m}L'_{00}\left[\frac{1}{R'_{00}(R'_{00}-z'_{00})}-\frac{y^2}{R'^3_{00}(R'_{00}-z'_{00})}-\frac{y^2}{R'^2_{00}(R'_{00}-z'_{00})^2}\right]$$

$$+\sum_{i=1}^{3}\omega'_{im}\sum_{j=1}^{3}L'_{ij}\left[\frac{1}{R'_{ij}(R'_{ij}-z'_{ij})}-\frac{x^2}{R'^3_{ij}(R'_{ij}-z'_{ij})}-\frac{x^2}{R'^2_{ij}(R'_{ij}-z'_{ij})^2}\right]$$

$$\tau'_{ym} = -\omega'_{0m}xy\left\{L'_{00}\left[\frac{1}{R'^3_{00}(R'_{00}-z'_{00})}+\frac{1}{R'^2_{00}(R'_{00}-z'_{00})^2}\right]\right\}$$

$$-xy\sum_{i=1}^{3}\omega'_{im}\sum_{j=1}^{3}L'_{ij}\left[\frac{1}{R'^3_{ij}(R'_{ij}-z'_{ij})}+\frac{1}{R'^2_{ij}(R'_{ij}-z'_{ij})^2}\right]$$

$$\sigma'_m = x\sum_{i=1}^{3}\vartheta'_{im}\sum_{j=1}^{3}\frac{L'_{ij}}{R'^3_{ij}}. \tag{54}$$

Similarly, it follows from the boundary conditions on the interface stated as eqns (8) that

$$D_0+D_{00} = L'_{00} \tag{55}$$

$$D_i+\sum_{j=1}^{3}D_{ji} = \sum_{j=1}^{3}L'_{ji} \tag{56}$$

$$\alpha_{im}D_i-\sum_{j=1}^{3}\alpha_{jm}D_{ji} = \sum_{j=1}^{3}\alpha'_{jm}L'_{ji} \tag{57}$$

$$\omega_{01}(D_{00}-D_0) = -\omega'_{01}L'_{00} \tag{58}$$

$$-\omega_{i1}D_i+\sum_{j=1}^{3}\omega_{j1}D_{ji} = -\sum_{j=1}^{3}\omega'_{j1}L'_{ji} \tag{59}$$

$$\vartheta_{im}D_i+\sum_{j=1}^{3}\vartheta_{jm}D_{ji} = \sum_{j=1}^{3}\vartheta'_{jm}L'_{ji} \tag{60}$$

where $i = 1, 2, 3$, $m = 1, 2$; D_{00}, D_{ij}, L'_{00} and L'_{ij} $(i,j = 1,2,3)$ are arbitrary constants that can be readily determined by the above algebraic equations.

4. SOLUTIONS FOR SEMI-INFINITE TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIA

From the above results, we can obtain the extension of Mindlin results in elasticity and of Lorentz results easily.

4.1. *Extension of Mindlin results*

For this case, the terms on the right-hand side of eqn (8b) are equal to zero, that is, $\sigma'_{zx} = 0$, $\tau'_{xz} = 0$, $\tau'_{yz} = 0$ and $D'_z = 0$. By setting the right-hand side of eqns (33) and (34) to zero, respectively, A_{ij} can be readily determined by the 9 algebraic equations. Similarly, we can determine D_{ij} by eqns (58)–(60), so they can be conveniently solved as follows:

$$
\begin{aligned}
A_{11} &= A_1(\vartheta_{21}\vartheta_{32}\omega_{11}-\vartheta_{22}\vartheta_{31}\omega_{11}-\vartheta_{12}\vartheta_{31}\omega_{21}+\vartheta_{11}\vartheta_{32}\omega_{21}+\vartheta_{12}\vartheta_{21}\omega_{31}-\vartheta_{11}\vartheta_{22}\omega_{31})/d_a\\
A_{21} &= 2A_1(\vartheta_{12}\vartheta_{31}-\vartheta_{11}\vartheta_{32})\omega_{11}/d_a\\
A_{31} &= 2A_1(\vartheta_{11}\vartheta_{22}-\vartheta_{12}\vartheta_{21})\omega_{11}/d_a\\
A_{12} &= 2A_2(\vartheta_{21}\vartheta_{32}-\vartheta_{22}\vartheta_{31})\omega_{21}/d_a\\
A_{22} &= A_2(\vartheta_{22}\vartheta_{31}\omega_{11}-\vartheta_{21}\vartheta_{32}\omega_{11}+\vartheta_{12}\vartheta_{31}\omega_{21}-\vartheta_{11}\vartheta_{32}\omega_{21}+\vartheta_{12}\vartheta_{21}\omega_{31}-\vartheta_{11}\vartheta_{22}\omega_{31})/d_a\\
A_{32} &= 2A_2(\vartheta_{11}\vartheta_{22}-\vartheta_{12}\vartheta_{21})\omega_{21}/d_a\\
A_{13} &= 2A_3(\vartheta_{21}\vartheta_{32}-\vartheta_{22}\vartheta_{31})\omega_{31}/d_a\\
A_{23} &= 2A_3(\vartheta_{12}\vartheta_{31}-\vartheta_{11}\vartheta_{32})\omega_{31}/d_a\\
A_{33} &= A_3(\vartheta_{22}\vartheta_{31}\omega_{11}-\vartheta_{21}\vartheta_{32}\omega_{11}-\vartheta_{12}\vartheta_{31}\omega_{21}+\vartheta_{11}\vartheta_{32}\omega_{21}-\vartheta_{12}\vartheta_{21}\omega_{31}+\vartheta_{11}\vartheta_{22}\omega_{31})/d_a\\
d_a &= \vartheta_{22}\vartheta_{31}\omega_{11}-\vartheta_{21}\vartheta_{32}\omega_{11}-\vartheta_{12}\vartheta_{32}\omega_{21}+\vartheta_{11}\vartheta_{32}\omega_{21}+\vartheta_{12}\vartheta_{21}\omega_{31}-\vartheta_{11}\vartheta_{22}\omega_{31} \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
D_{00} &= D_0\\
D_{ji} &= D_i A_{ji}/A_i, \quad (i,j=1,2,3). \qquad (62)
\end{aligned}
$$

4.2. *Extension of Lorentz results*

For this case, the terms on the right-hand side of eqn (8a) are equal to zero, that is, $u' = 0$, $v' = 0$, $w'_m = 0$ and $\phi' = 0$. We can set the right-hand sides of eqns (31), (32) and (55)–(57) to zero. We can hence determine the constants of A_{ij}, and G_{ij}.

5. CONCLUSIONS

1. The above solutions are restricted to the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ and $s'_1 \neq s'_2 \neq s'_3 \neq s'_1$. For other cases of multiple roots, the general solutions have not been discussed in Ding *et al.* (1996), their expressions are given in Appendix A. The fundamental solutions for infinite media and the Green Functions for the two-phase media are listed in Appendix B.

2. If the two half-spaces are in smooth contact, the boundary conditions on the interface are:

$$
\begin{aligned}
w_m &= w'_m, \quad \sigma_m = \sigma'_m, \quad (m=1,2)\\
\tau_{xz} &= \tau'_{xz} = \tau_{yz} = \tau'_{yz} = 0. \qquad (63)
\end{aligned}
$$

Following the same approach as above, the arbitrary constants A_{ij}, A'_{ij}, D_{00}, D_{ij}, L'_{00} and L'_{ij} can also be determined by the boundary conditions stated as eqn (63). We thus obtain the solutions for two smooth contact semi-infinite transversely isotropic piezoelectric solids.

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APPENDIX A: THE GENERAL SOLUTIONS FOR THE CASES OF MULTIPLE ROOTS

The determinant of the characteristic eqn (c7) in Suo *et al.* (1992) is a third-order polynomial in p^2, it is easy to verify that the equation can be written in the following form

$$-ap^6 - bp^4 - cp^2 - d = 0 \tag{A-1}$$

where a, b, c and d are the functions of c_{ij}, e_{ij}, ε_{ij}; they are defined in Ding *et al.* (1996). It is obvious that if we set $p^2 = -s^2$, the above equation becomes

$$as^6 - bs^4 + cs^2 - d = 0, \tag{A-2}$$

which is just the eqn (32) in Ding *et al.* (1996). According to Suo *et al.* (1992), eqn (A-2) can be written

$$(s^2 - \alpha^2)(s^4 - 2\rho\xi^2 s^2 + \xi^4) = 0. \tag{A-3}$$

For stable materials, they are restricted so that

$$\alpha > 0, \quad \xi > 0, \quad \rho > -1. \tag{A-4}$$

From eqn (A-3), we have

$$s_1 = \alpha$$
$$s_2 = \xi\left[\left(\frac{\rho+1}{2}\right)^{1/2} + \left(\frac{\rho-1}{2}\right)^{1/2}\right]$$
$$s_3 = \xi\left[\left(\frac{\rho+1}{2}\right)^{1/2} - \left(\frac{\rho-1}{2}\right)^{1/2}\right]. \tag{A-5}$$

These expressions ensure that $\mathrm{Re}\, s_i > 0$ $(i = 1, 2, 3)$.

From eqns (A-5) and (A-4), we can get some useful attributes of s_i: when $\rho > 1$, $s_2 > 0$ and $s_3 > 0$; when $\rho = 1$, $s_2 = s_3 = \xi > 0$; when $-1 < \rho < 1$, s_2 and s_3 are a conjugate pair, and it is impossible to change them into pure imaginary roots; when $\rho = 1$ and $\xi = \alpha$, $s_1 = s_2 = s_3 = \alpha$.

The general solutions can also be expressed in terms of some "harmonic" functions for the cases of multiple roots of s_i. After performing similar derivations as what Ding *et al.* (1996) have done for the case of $s_1 \neq s_2 \neq s_3 \neq s_1$, we have

1. for the case of $s_1 \neq s_2 = s_3$

$$u = -\frac{\partial \psi_0}{\partial y} + \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x} + z_2 \frac{\partial \psi_3}{\partial x}$$
$$v = \frac{\partial \psi_0}{\partial x} + \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial y} + z_2 \frac{\partial \psi_3}{\partial y}$$

$$w_m = \alpha_{1m}\frac{\partial\psi_1}{\partial z_1} + \alpha_{2m}\frac{\partial\psi_2}{\partial z_2} + \alpha_{2m}z_2\frac{\partial\psi_3}{\partial z_2} + \alpha_{4m}\psi_3, \quad (m = 1, 2) \tag{A-6}$$

where

$$\alpha_{41} = \frac{2(2c_{44}\varepsilon_{33}s_2^2 - m_3)s_2 - (m_1 - 3m_2s_2^2)\alpha_{21}}{m_1 - m_2s_2^2}$$

$$\alpha_{42} = \frac{2(2c_{44}\varepsilon_{33}s_2^2 - m_4)s_2 - (m_1 - 3m_2s_2^2)\alpha_{22}}{m_1 - m_2s_2^2} \tag{A-7}$$

2. for the case of $s_1 = s_2 = s_3$

$$u = -\frac{\partial\psi_0}{\partial y} + \frac{\partial\psi_1}{\partial x} + z_1\frac{\partial\psi_2}{\partial x} + z_1^2\frac{\partial^2\psi_3}{\partial x\,\partial z_1}$$

$$v = \frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_1}{\partial y} + z_1\frac{\partial\psi_2}{\partial y} + z_1^2\frac{\partial^2\psi_3}{\partial y\,\partial z_1}$$

$$w_m = \alpha_{1m}\frac{\partial\psi_1}{\partial z_1} + \alpha_{1m}z_1\frac{\partial\psi_2}{\partial z_1} + \alpha_{4m}\psi_2 + \alpha_{2m}z_1^2\frac{\partial^2\psi_3}{\partial z_1^2} + 2\alpha_{4m}z_1\frac{\partial\psi_3}{\partial z_1} + \alpha_{3m}\psi_3, \quad (m = 1, 2) \tag{A-8}$$

where

$$\alpha_{51} = \frac{2[3m_2(\alpha_{11} + \alpha_{41})s_1^2 - m_1\alpha_{41} + (6c_{44}\varepsilon_{33}s_1^2 - m_3)s_1]}{m_1 - m_2s_1^2}$$

$$\alpha_{52} = \frac{2[3m_2(\alpha_{12} + \alpha_{42})s_1^2 - m_1\alpha_{42} + (6c_{44}e_{33}s_1^2 - m_4)s_1]}{m_1 - m_2s_1^2}. \tag{A-9}$$

The functions ψ_i, appearing in eqns (A-6) and (A-8), are also satisfied with eqn (3).

APPENDIX B: SOLUTIONS FOR THE CASES OF MULTIPLE ROOTS

B1. Solutions to the problem of combination of point force P *in* z *direction and point charge* Q

a. When $s_1 \neq s_2 \neq s_3 \neq s_1$, in the half space of $z \geqslant 0$, the components of mechanical displacements, stresses, electric potential and electric displacement are expressed as eqn (28). In the half-space of $z \leqslant 0$, the components are expressed differently in three cases:

a1. when $s'_1 \neq s'_2 \neq s'_3 \neq s'_1$, they have been solved;

a2. when $s'_1 \neq s'_2 = s'_3$, assuming

$$\psi'_i = \sum_{j=1}^{3} B'_{ij}\ln(R'_{ij} - z'_{ij}), \quad (i = 1, 2); \quad \psi'_3 = \sum_{j=1}^{3}\frac{B'_{3j}}{R'_{2j}}; \tag{B-1}$$

a3. when $s'_1 = s'_2 = s'_3$, assuming

$$\psi'_1 = \sum_{j=1}^{3} C'_{ij}\ln(R'_{ij} - z'_{ij}), \quad \psi'_i = \sum_{j=1}^{3}\frac{C'_{ij}}{R'_{ij}}, \quad (i = 2, 3). \tag{B-2}$$

b. When $s_1 \neq s_2 = s_3$, assuming

$$\psi_i = \text{sign}(z-h)B_i\ln(R_{ii} + s_i|z-h|) + \sum_{j=1}^{2} B_{ij}\ln(R_{ij} + z_{ij}) + \frac{B_{i3}}{R_{i2}}, \quad (i = 1, 2)$$

$$\psi_3 = \frac{B_3}{R_{22}} + \sum_{j=1}^{2}\frac{B_{3i}}{R_{2i}} + \frac{B_{33}z_{22}}{R_{22}^3} \tag{B-3}$$

b1. when $s'_1 \neq s'_2 \neq s'_3 \neq s'_1$, assuming

$$\psi'_i = \sum_{j=1}^{2} D'_{ij}\ln(R'_{ij} - z'_{ij}) + \frac{D'_{i3}}{R'_{i2}}, \quad (i = 1, 2, 3); \tag{B-4}$$

b2. when $s'_1 \neq s'_2 \neq s'_3$, assuming

$$\psi_i' = \sum_{j=1}^{2} E_{ij}' \ln(R_{ij}' - z_{ij}') + \frac{E_{i3}'}{R_{i2}'}, \quad (i = 1, 2), \quad \psi_j' = \sum_{j=1}^{2} \frac{E_{3j}'}{R_{2j}'} + \frac{E_{33}' z_{22}'}{R_{22}'^3}; \tag{B-5}$$

b3. when $s_1' = s_2' = s_3'$, assuming

$$\psi_1' = \sum_{j=1}^{2} G_{ij}' \ln(R_{ij}' - z_{ij}') + \frac{G_{13}'}{R_{12}'} \quad \psi_i' = \sum_{j=1}^{2} \frac{G_{ij}'}{R_{ij}'} + \frac{G_{i3}' z_{i2}'}{R_{12}'^3} \quad (i = 2, 3). \tag{B-6}$$

c. When $s_1 = s_2 = s_3$, assuming

$$\psi_1 = \operatorname{sign}(z-h) C_1 \ln(R_{11} + S_1|z-h|) + C_{11} \ln(R_{11} + z_{11}) + \frac{C_{12}}{R_{11}} + \frac{C_{13} z_{11}}{R_{11}^3}$$

$$\psi_i = \frac{C_i}{R_{11}} + \frac{C_{i1}}{R_{11}} + \frac{C_{i2} z_{11}}{R_{11}^3} + C_{i3}\left(\frac{3z_{11}^2}{R_{11}^5} - \frac{1}{R_{11}^3}\right), \quad (i = 2, 3) \tag{B-7}$$

c1. when $s_1' \neq s_2' \neq s_3' \neq s_1'$, assuming

$$\psi_i' = H_{i1}' \ln(R_{i1}' - z_{i1}') + \frac{H_{i2}'}{R_{i1}'} + \frac{H_{i3}' z_{i1}'}{R_{i1}'^3} \quad (i = 1, 2, 3); \tag{B-8}$$

c2. when $s_1' \neq s_2' = s_3'$, assuming

$$\psi_i' = J_{i1}' \ln(R_{i1}' - z_{i1}') + \frac{J_{i2}'}{R_{i1}'} + \frac{J_{i3}' z_{i1}'}{R_{i1}'^3}, \quad (i = 1, 2)$$

$$\psi_3' = \frac{J_{31}'}{R_{21}'} + \frac{J_{32}' z_{21}'}{R_{21}'^3} + J_{33}'\left(\frac{3z_{21}'^2}{R_{21}'^5} - \frac{1}{R_{21}'^3}\right); \tag{B-9}$$

c3. when $s_1' = s_2' = s_3'$, assuming

$$\psi_1' = K_{11}' \ln(R_{11}' - z_{11}') + \frac{K_{12}'}{R_{11}'} + \frac{K_{13}' z_{11}'}{R_{11}'^3}$$

$$\psi_i' = \frac{K_{i1}'}{R_{11}'} + \frac{K_{i2}' z_{11}'}{R_{11}'^3} + K_{i3}'\left(\frac{3z_{11}'^2}{R_{11}'^5} - \frac{1}{R_{11}'^3}\right), \quad (i = 2, 3). \tag{B-10}$$

B2. Solutions to the problem of point force T *in* x *direction*

d. When $s_1 \neq s_2 \neq s_3 \neq s_1$, in the half-space of $z \geqslant 0$, the components of displacements, stresses, electric potential and electric displacement are expressed as in eqn (52). In the half-space of $z \leqslant 0$, the components are expressed differently in three cases:

d1. when $s_1' \neq s_2' \neq s_3' \neq '_1$, they have been solved;

d2. when $s_1' \neq s_2' = s_3'$, assuming

$$\psi_0' = \frac{M_{00}' y}{R_{00}' - z_{00}'}, \quad \psi_i' = \sum_{j=1}^{3} \frac{M_{ij}' x}{R_{ij}' - z_{ij}'}, \quad (i = 1, 2); \quad \psi_3' = \sum_{j=1}^{3} \frac{M_{3j}' x}{R_{1j}'(R_{2j}' - z_{2j}')}; \tag{B-11}$$

d3. when $s_1' = s_2' = s_3'$, assuming

$$\psi_0' = \frac{N_{00}' y}{R_{00}' - z_{00}'}; \quad \psi_1' = \sum_{j=1}^{3} \frac{N_{1j}' x}{R_{1j}' - z_{1j}'}; \quad \psi_i' = \sum_{j=1}^{3} \frac{N_{ij}' x}{R_{1j}'(R_{1j}' - z_{1j}')}, \quad (i = 2, 3). \tag{B-12}$$

e. When $s_1 \neq s_2 = s_3$, assuming

$$\psi_0 = \frac{E_0 y}{R_{00} + s_0|z-h|} + \frac{E_{00} y}{R_{00} + z_{00}}$$

$$\psi_i = \frac{E_i x}{R_{ii} + s_i|z-h|} + \sum_{j=1}^{2} \frac{E_{ij} x}{R_{ij} + z_{ij}} + \frac{E_{i3} x}{R_{i2}(R_{i2} + z_{i2})}, \quad (i = 1, 2)$$

$$\psi_3 = \operatorname{sign}(z-h) \frac{E_3 x}{R_{22}(R_{22} + s_2|z-h|)} + \sum_{j=1}^{2} \frac{E_{3j} x}{R_{2j}(R_{2j} + z_{2j})} + \frac{E_{33} x}{R_{22}^3} \tag{B-13}$$

e1. when $s_1' \neq s_2' \neq s_3' \neq s_1'$, assuming

$$\psi'_0 = \frac{O'_{00}}{R'_{00}+z'_{00}}$$

$$\psi'_i = \sum_{j=1}^{2} \frac{O'_{ij}x}{R'_{ij}-z'_{ij}} + \frac{O'_{i3}x}{R'_{i2}(R'_{i2}-z'_{i2})}, \quad (i = 1,2,3); \tag{B-14}$$

e2. when $s'_1 \neq s'_2 = s'_3$, assuming

$$\psi'_0 = \frac{P'_{00}y}{R'_{00}+z'_{00}}$$

$$\psi'_i = \sum_{j=1}^{2} \frac{P'_{ij}x}{R'_{ij}-z'_{ij}} + \frac{P'_{i3}x}{R'_{i2}(R'_{i2}-z'_{i2})}, \quad (i = 1,2)$$

$$\psi'_3 = \sum_{j=1}^{2} \frac{P'_{3j}x}{R'_{2j}(R'_{2j}-z'_{2j})} + \frac{P'_{33}x}{R'^3_{22}}; \tag{B-15}$$

e3. when $s'_1 = s'_2 = s'_3$, assuming

$$\psi'_0 = \frac{Q'_{00}y}{R'_{00}-z'_{00}}$$

$$\psi'_1 = \sum_{j=1}^{2} \frac{Q'_{1j}x}{R'_{1j}-z'_{1j}} + \frac{Q'_{13}x}{R'_{12}(R'_{12}-z'_{12})}$$

$$\psi'_i = \sum_{j=1}^{2} \frac{Q'_{ij}x}{R'_{1j}(R'_{1j}-z'_{1j})} + \frac{Q'_{i3}x}{R'^3_{12}}, \quad (i = 2,3). \tag{B-16}$$

f. When $s_1 = s_2 = s_3$, assuming

$$\psi_0 = \frac{G_0 y}{R_{00}+s_0|z-h|} + \frac{G_{00}y}{R_{00}+z_{00}}$$

$$\psi_1 = \frac{G_1 x}{R_{11}+s_1|z-h|} + \frac{G_{11}x}{R_{11}+z_{11}} + \frac{G_{12}x}{R_{11}(R_{11}+z_{11})} + \frac{G_{13}x}{R^3_{11}}$$

$$\psi_i = \mathrm{sign}(z-h)\frac{G_i x}{R_{11}(R_{11}+s_1|z-h|)} + \frac{G_{i1}x}{R_{11}(R_{11}+z_{11})} + \frac{G_{i2}x}{R^3_{11}} + \frac{G_{i3}xz_{11}}{R^5_{11}}, \quad (i = 2,3) \tag{B-17}$$

f1. when $s'_1 \neq s'_2 \neq s'_3 \neq s'_1$, assuming

$$\psi'_0 = \frac{S'_{00}y}{R'_{00}-z'_{00}}$$

$$\psi'_i = \frac{S'_{i1}x}{R'_{i1}-z'_{i1}} + \frac{S'_{i2}x}{R'_{i1}(R'_{i1}-z'_{i1})} + \frac{S'_{i3}x}{R'^3_{i1}}, \quad (i = 1,2,3); \tag{B-18}$$

f2. when $s'_1 \neq s'_2 = s'_3$, assuming

$$\psi'_0 = \frac{T'_{00}y}{R'_{00}-z'_{00}}$$

$$\psi'_i = \frac{T'_{i1}x}{R'_{i1}-z'_{i1}} + \frac{T'_{i2}x}{R'_{i1}(R'_{i1}-z'_{i1})} + \frac{T'_{i3}x}{R'^3_{i1}}, \quad (i = 1,2)$$

$$\psi'_3 = \frac{T'_{31}x}{R'_{21}(R'_{21}-z'_{21})} + \frac{T'_{32}x}{R'^3_{21}} + \frac{T'_{33}xz'_{21}}{R'^5_{21}}; \tag{B-19}$$

f3. when $s'_1 = s'_2 = s'_3$, assuming

$$\psi'_0 = \frac{\pi'_{00}y}{R'_{00}-z'_{00}}$$

$$\psi'_1 = \frac{\pi'_{11}x}{R'_{11}-z'_{11}} + \frac{\pi'_{12}x}{R'_{11}(R'_{11}+z'_{11})} + \frac{\pi'_{13}x}{R'^3_{11}}$$

$$\psi'_i = \frac{\pi'_{i1}x}{R'_{11}(R'_{11}-z'_{11})} + \frac{\pi'_{i2}x}{R'^3_{11}} + \frac{\pi'_{i3}xz'_{11}}{R'^5_{11}}, \quad (i = 2,3). \tag{B-20}$$

Substituting eqns (B-1)–(B-20) into the general solutions of eqns (2) or (A-6) or (A-8), the expressions of mechanical displacement and electric potential are obtained, then by the constitutive relations the expressions of stresses and electric displacement are obtained. Consideration of the continuity of displacement components u, v and stress components σ_x, σ_y, τ_{xy} on $z = h$, and of the equilibrium eqns (22)–(23) will lead to the following equations to determine the constants B_i and C_i

$$\text{b.}\quad B_1+B_2=0\quad 4\pi\vartheta_{11}B_1+4\pi(\vartheta_{21}B_2+\vartheta_{41}B_3)-P=0\quad 4\pi\vartheta_{12}B_1+4\pi(\vartheta_{22}B_2+\vartheta_{42}B_3)+Q=0 \tag{B-21}$$

$$\text{c.}\quad C_1=0\quad 4\pi(\vartheta_{11}C_1+\vartheta_{41}C_2+\vartheta_{51}C_3)-P=0\quad 4\pi(\vartheta_{12}C_1+\vartheta_{42}C_2+\vartheta_{32}C_3)+Q=0 \tag{B-22}$$

where

$$\begin{aligned}
\vartheta_{41} &= [c_{33}(\alpha_{21}+\alpha_{41})+e_{33}(\alpha_{22}+\alpha_{42})]s_2\\
\vartheta_{42} &= [e_{33}(\alpha_{21}+\alpha_{41})-\varepsilon_{33}(\alpha_{22}+\alpha_{42})]s_2\\
\vartheta_{51} &= [c_{33}(2\alpha_{41}+\alpha_{51})+e_{33}(2\alpha_{42}+\alpha_{32})]s_1\\
\vartheta_{52} &= [e_{33}(2\alpha_{41}+\alpha_{51})-\varepsilon_{33}(2\alpha_{42}+\alpha_{52})]s_1.
\end{aligned} \tag{B-23}$$

Consideration of the continuity of w_m, τ_{xm}, τ_{ym} on $z = h$ and the equilibrium eqn (48), the following equations are obtained to determine the constants E_i and G_i

$$\text{e.}\quad \alpha_{1m}E_1+\alpha_{2m}E_2-\alpha_{4m}E_3=0,\quad (m=1,2) \tag{B-24}$$

$$\omega_{0m}E_0+\sum_{i=1}^{2}\omega_{im}E_i-\omega_{4m}E_3=0,\quad (m=1,2) \tag{B-25}$$

$$c_{44}s_0E_0-\omega_{11}E_1-\omega_{21}E_2+\omega_{41}E_3+\frac{T}{2\pi}=0 \tag{B-26}$$

where

$$\begin{aligned}
\omega_{41} &= c_{44}s_2+c_{44}\alpha_{41}+e_{15}\alpha_{42}\\
\omega_{42} &= e_{15}s_2+e_{15}\alpha_{41}-\varepsilon_{11}\alpha_{42}.
\end{aligned} \tag{B-27}$$

Equation (B-25) can be simplified by virtue of eqns (7), (B-27) and (B-24)

$$s_0E_0+s_1E_1+s_2E_2-s_2E_3=0. \tag{B-28}$$

The constants E_i ($i = 0, 1, 2, 3$) can be determined by solving the algebra equations (B-24), (B-26) and (B-28).

$$\text{f.}\quad \alpha_{1m}G_1-\alpha_{4m}G_2-\alpha_{5m}G_3=0,\quad (m=1,2) \tag{B-29}$$

$$\omega_{0m}G_0+\omega_{m1}G_1-\omega_{4m}G_2-\omega_{5m}G_3=0,\quad (m=1,2) \tag{B-30}$$

$$c_{44}s_0G_0-\omega_{11}G_1+\omega_{41}G_2+\omega_{51}G_3+\frac{T}{2\pi}=0 \tag{B-31}$$

where

$$\begin{aligned}
\omega_{51} &= c_{44}\alpha_{51}+e_{15}\alpha_{52}\\
\omega_{52} &= e_{15}\alpha_{51}-\varepsilon_{11}\alpha_{52}.
\end{aligned} \tag{B-32}$$

Equation (B-30) can be simplified by virtue of eqns (7), (B-29) and (B-32)

$$s_0G_0+s_1(G_1-G_2)=0. \tag{B-33}$$

The constants G_i ($i = 0, 1, 2, 3$) can be determined by solving the algebra eqns (B-29), (B-31) and (B-33).

After the constants of the fundamental solutions for infinite media are obtained, the constants of A_{ij} and B'_{ij}, A_{ij} and C'_{ij}, …, G_{ij} and π'_{ij} can be determined in pairs by the boundary conditions eqn (8). The control equations for each pair of constants are similar to eqns (31)–(34) and (55)–(60). With *Mathematica*, such generation of equations is easy and much more reliable.